ON SUFFICIENT CONDITIONS FOR AN OPTIMUM

(O DOSTATOCHNYKH USLOVIIAKH OPTIMAL'NOSTI)

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In this note there are established sufficient conditions for an optimum in a particular case. It is shown that a part of these conditions is a condition of the maximum principle of Pontriagin [1].

Let us consider the system of differential equations

$$\dot{x}_i = f_i(x_1, \ldots, x_n; u_1, \ldots, u_r; t) \qquad (i = 1, \ldots, n) \tag{1}$$

which describes an automatic control process. Here $x_1(t), \ldots, x_n(t)$ are parameters of the object, $u_1(t), \ldots, u_r(t)$ are positions of the controlling elements.

It is assumed that the functions f_i are continuous and bounded for all their arguments, and that they have continuous partial derivatives of the first order with respect to $x_1, \ldots, x_n; u_1, \ldots, u_r$. It is also assumed that $u_1, \ldots, u_r$ are continuous and satisfy the inequalities

$$g_j(u_1, \ldots, u_r) \leqslant 0 \qquad (j = 1, \ldots, m) \tag{2}$$

We shall refer to them in the sequel as "admissible controls". Suppose that the system (1) has the initial conditions

$$x_i(t_0) = x_i^\circ \qquad (i = 1, \ldots, n) \tag{3}$$

where the x_i° are given quantities.

In [2] it was shown that the problem of optimum control can be reduced to the consideration of the system (1) (we shall assume that the new variable has already been introduced into (1)) for which it is required to select $u_1, \ldots, u_r$ from the admissible controls so that they will transform the system (1), at the point $x_0(t_0) = x^\circ$ in the phase space, so that the sum

$$S = \sum_{i=1}^{n} c_i x_i(T) \qquad (c_i = \text{const}) \tag{4}$$

will take on a maximum (or minimum) value at the given instant of time $t = T$.

Let us consider the case when no restrictions are imposed on $x_1, \ldots, x_n$ at $t = T$. In [3] there is given a formula for the increment of the value of the functional S when the control is changed:

$$\Delta S(T) = \sum_{i=1}^{n} c_i \delta x_i(T)$$
$$= -\int_{t_0}^{T} \left\{ \left[\sum_{i=1}^{n} \left(\dot{\lambda}_i + \sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial x_i} \right) \delta x_i + \sum_{k=1}^{r} \sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial u_k} \delta u_k \right] + \sum_{i=1}^{n} \lambda_i \varepsilon_i \right\} dt \qquad (5)$$

Here the $\lambda_i(t)$ are multipliers, and the ϵ_i are infinitesimals of higher order than the first. Under the given conditions we have

$$\sum_{i=1}^{n} \lambda_i \varepsilon_i = \sum_{i=1}^{n} \lambda_i \left(\sum_{j=1}^{n} \delta x_j \frac{\partial}{\partial x_j} + \sum_{k=1}^{r} \delta u_k \frac{\partial}{\partial u_k} \right)^2 f_i + \sum_{i=1}^{n} \lambda_i \eta_i \qquad (6)$$

where the η_i are infinitesimals of order higher than the first.

Let us introduce the function H of the variables $x_1, \ldots, x_n$; $\lambda_1, \ldots, \lambda_n$; $u_1, \ldots, u_r$; t,

$$H = \sum_{i=1}^{n} \lambda_i f_i \qquad (7)$$

Then Expression (6) will have the form

$$\sum_{i=1}^{n} \lambda_i \varepsilon_i = L_2 H + \sum_{i=1}^{n} \lambda_i \eta_i \qquad (8)$$

where L_2 is an operator.

Obviously, $L_2 H$ is a quadratic form in the variations δx_j $(j = 1, \ldots, n)$ and δu_k $(k = 1, \ldots, r)$.

We shall now pass to the consideration of the conditions for an optimum of the control $u_1, \ldots, u_r$. For the sake of definiteness, let us assume that $u_1, \ldots, u_r$ yields a maximum of the functional $S(T)$. Then for an arbitrary change δu_k $(k = 1, \ldots, r)$ we have

$$\Delta S(T) \leqslant 0$$

We shall denote the linear part of the increment $\Delta S(T)$ by $\delta S(T)$.

Since $\delta S(T) = 0$, it follows from [3] that a necessary condition for an optimum is given by

$$\frac{\partial H}{\partial u_k} = 0 \qquad (k = 1, \ldots, r) \tag{9}$$

provided that

$$\lambda_i = -\frac{\partial H}{\partial x_i}, \quad \lambda_i(T) = -c_i \qquad (i = 1, \ldots, n) \tag{10}$$

Under the conditions (9) and (10) it follows from (5) and (8) that the sign of $\Delta S(T)$ is completely determined by the sign of the quadratic form L_2H. Therefore, if $L_2H > 0$, the condition that $\delta S(T) \leqslant 0$ will be satisfied. It is not difficult to state the necessary and sufficient conditions for the positive-definiteness of the quadratic form L_2H when $t_0 \leqslant t \leqslant T$:

$$D_1 = \frac{\partial^2 H}{\partial u_1^2} > 0, \quad D_2 = \begin{vmatrix} \frac{\partial^2 H}{\partial u_1^2} & \frac{\partial^2 H}{\partial u_1 \partial u_2} \\ \frac{\partial^2 H}{\partial u_2 \partial u_1} & \frac{\partial^2 H}{\partial u_2^2} \end{vmatrix} > 0, \quad D_r = \begin{vmatrix} \frac{\partial^2 H}{\partial u_1^2} & \cdots & \frac{\partial^2 H}{\partial u_1 \partial u_r} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 H}{\partial u_r \partial u_1} & \cdots & \frac{\partial^2 H}{\partial u_r^2} \end{vmatrix} > 0$$

$$D_{r+1} = \begin{vmatrix} \frac{\partial^2 H}{\partial u_1^2} & \cdots & \frac{\partial^2 H}{\partial u_1 \partial u_r} & \frac{\partial^2 H}{\partial u_1 \partial x_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 H}{\partial x_1 \partial u_1} & \cdots & \frac{\partial^2 H}{\partial x_1 \partial u_r} & \frac{\partial^2 H}{\partial x_1^2} \end{vmatrix} > 0, \ldots, \quad D_{r+n} = \begin{vmatrix} \frac{\partial^2 H}{\partial u_1^2} & \cdots & \frac{\partial^2 H}{\partial u_1 \partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 H}{\partial x_n \partial u_1} & \cdots & \frac{\partial^2 H}{\partial x_n^2} \end{vmatrix} > 0 \tag{11}$$

These are sufficient conditions for an optimum under the conditions (9) and (10).

If the first r conditions of (11) are satisfied then

$$\delta^2 H = \sum_{s=1}^{r} \sum_{k=1}^{r} \frac{\partial^2 H}{\partial u_k \partial u_s} \delta u_k \delta u_s > 0$$

On the other hand, the increment ΔH of the function H due to a change in the control will have the form

$$\Delta H = \delta H + \delta^2 H + \eta, \quad \delta H = \sum_{k=1}^{r} \sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial u_k} \delta u_k = \sum_{k=1}^{r} \frac{\partial H}{\partial u_k} \delta u_k = 0$$

Here η is an infinitesimal of higher order. This shows that the sign of ΔH is completely determined by the sign of $\delta^2 H$. Here $\delta^2 H > 0$. Therefore, $\Delta H > 0$. This means that the optimum control corresponding to the maximum value of the functional $S(T)$ yields a minimum value of the function H. This fact constitutes the principle of Pontriagin.

In this manner it can be seen that in the given case, when the control lies entirely inside the region (2), the condition of the principle of

Pontriagin is a part of the conditions (9) and (10).

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